A CLASS OF COMPLETELY INTEGRABLE QUANTUM SYSTEMS ASSOCIATED WITH CLASSICAL ROOT SYSTEMS

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Dedicated to Professor Gerrit van Dijk on the occasion of his sixty fifth birthday

Abstract. We classify the completely integrable systems associated with classical root systems whose potential functions are meromorphic at an infinite point.

1. Introduction

A Shrödinger operator

$$P=\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+R(x) \tag{1.1}$$

with the potential function $R(x)$ of n variables $x=(x_1,\dots,x_n)$ is called *completely integrable* if there exist n differential operators $P_1,\dots,P_n$ such that

$$\begin{cases}[P_i,P_j]=0 \qquad (1\le i<j\le n),\\ P\in\mathbb{C}[P_1,\dots,P_n],\\ P_1,\dots,P_n \text{ are algebraically independent.}\end{cases} \tag{1.2}$$

In this note P is called to be completely integrable *of type* B_n or *of classical type* if P_k and $R(x)$ in the above are of the forms

$$P_k=\sum_{j=1}^{n}\frac{\partial^{2k}}{\partial x_j^{2k}}+Q_k \qquad \text{with } \operatorname{ord}Q_k<\operatorname{ord}P_k, \tag{1.3}$$

$$R(x)=\sum_{1\le i<j\le n}\Bigl(u_{ij}^-(x_i-x_j)+u_{ij}^+(x_i+x_j)\Bigr)+\sum_{k=1}^{n}v_k(x_k). \tag{1.4}$$

Here $u_{ij}^{\pm}$ and v_k are functions of one variable.

The systems of differential operators satisfied by the radial parts of zonal spherical functions or Whittaker functions on Riemannian symmetric spaces of the noncompact and classical type, Heckman-Opdam's hypergeometric equations (cf. [HO]), Calogero-Moser and Sutherland systems for one dimensional quantum n-body problems (cf. [OP1], [OP2]) and Toda finite chains associated with (extended) classical Dynkin diagrams are their examples.

We remark that [Wa] proves that if the potential function $R(x)$ is locally defined and analytic, then the condition (1.2) with (1.3) assures (1.4) and moreover $R(x)$ is extended to a global meromorphic function on $\mathbb{C}^n$ except for a trivial case corresponding to Type A_1 in Theorem 4.8 (cf. [Oc] for type B_2 and [OS] in the invariant case).

[OOS], [OS], [O2] and [OO] determine this integrable system under the condition that P is B_n-invariant, namely, u_{ij}^+, u_{ij}^- and v_k are even functions and do not depend on i, j and k and $u_{ij}^+=u_{ij}^-$. On the other hand [Oc], [Ta] and [Wa] determine it if $R(x)$ has certain singularities.

We assume in this note that $R(x)$ is meromorphic at $t=0$ under the coordinate system

$$t_j = e^{-(x_j - x_{j+1})} \quad (j = 1, \ldots, n-1), \quad t_n = e^{-x_n} \tag{1.5}$$

and classify the Shrödinger operator (1.1) which allows a differential operator P_2 of the form (1.3) satisfying $PP_2 = P_2 P$. We note that the above examples with non-rational potential functions satisfy this assumption. In the first example this follows from the fact that the invariant differential operators on a Riemannian symmetric space have analytic extensions on a smooth compactification of the space (cf. [O1]).

Theorem 4.8 and Remark 4.7 in §4 determine $R(x)$, which is the main result of this note and proved by using §2 and §3. The result implies that the system is a suitable limit of the invariant quantum integrable system classified by [OOS] (cf. [I], [IM], [Ru], [vD] and [vD2]). Hence the integrals $P_1, \ldots, P_n$ will be calculated as suitable limits from the integrals given in [O2], which will be shown in another paper.

If $R(x)$ is analytic at $t=0$, we say that $R(x)$ has *regular singularity* at the infinite point $t=0$, which are also classified in Corollary 4.10.

In §3 the potential function $R(x)$ is determined when $n=2$.

In §2 we study the potential function $R(x)$ when $u_{ij}^{+} = v_k = 0$, which we call to be of type A_{n-1}.

2. Type A_{n-1} $(n \geq 3)$

In this section we study the Shrödinger operator

$$P = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{1 \leq i < j \leq n} \tilde{u}_{ij}(x_i - x_j) \tag{2.1}$$

which allows a differential operator

$$Q = \sum_{1 \leq i < i < k \leq n} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + S \qquad \text{with } \operatorname{ord} S < 3 \tag{2.2}$$

satisfying $[P, Q] = [\sum_{j=1}^{n} \frac{\partial}{\partial x_j}, Q] = 0$. Then the proof of [OOS, Proposition 4.2] implies that the existence of Q is equivalent to

$$\sum_{1 \leq i < j < k \leq n} U_{ijk} = 0 \tag{2.3}$$

with

(2.4)

$$\begin{aligned} U_{ijk} = {} & u_{jk}(t_j \cdots t_{k-1}) \Big(t_i \cdots t_{j-1} u'_{ij}(t_i \cdots t_{j-1}) + t_i \cdots t_{k-1} u'_{ik}(t_i \cdots t_{k-1}) \Big) \\ & + u_{ik}(t_i \cdots t_{k-1}) \Big(-t_i \cdots t_{j-1} u'_{ij}(t_i \cdots t_{j-1}) + t_j \cdots t_{k-1} u'_{jk}(t_j \cdots t_{k-1}) \Big) \\ & - u_{ij}(t_i \cdots t_{j-1}) \Big(t_i \cdots t_{k-1} u'_{ik}(t_i \cdots t_{k-1}) + t_j \cdots t_{k-1} u'_{jk}(t_j \cdots t_{k-1}) \Big) \end{aligned}$$

by putting $u_{ij}(e^{-y}) = \tilde{u}_{ij}(y)$. We assume that $R(x)$ is holomorphic for $0 < |t| \ll 1$ under the coordinate system (1.5) which corresponds to the expression

$$u_{ij}(s) = \sum_{\nu \in \mathbb{Z}} c_\nu^{ij} s^\nu \quad (c_0^{ij} = 0) \text{ converge for } 0 < |s| \ll 1. \tag{2.5}$$

We assume $c_0^{ij} = 0$ without loss of generality and expand (2.3) into the power series. Then the terms $(t_i \cdots t_{j-1})^p (t_j \cdots t_{k-1})^q$ with $p \neq 0$, $q \neq 0$, $p \neq q$ and $i < j < k$ appear only in U_{ijk} and therefore if $p \neq 0$, $q \neq 0$ and $p \neq q$, we have

$$c_q^{jk} p c_p^{ij} + c_{q-p}^{jk} p c_p^{ik} - c_q^{ik} (p-q) c_{p-q}^{ij} + c_p^{ik} (q-p) c_{q-p}^{jk} - c_{p-q}^{ij} q c_q^{ik} - c_p^{ij} q c_q^{jk} = 0$$

and hence

$$(p-q)c_p^{ij}c_q^{jk} - pc_{p-q}^{ij}c_q^{ik} + qc_{q-p}^{jk}c_p^{ik} = 0.$$

Denoting

$$U_{ij}(t) = \sum_{\nu\in\mathbb{Z}\setminus\{0\}} C_\nu^{ij}t^\nu \quad \text{with } c_\nu^{ij} = \nu C_\nu^{ij}, \tag{2.6}$$

we have

$$u_{ij}(t) = tU'_{ij}(t), \tag{2.7}$$

$$pq(p-q)\big(C_p^{ij}C_q^{jk} - C_{p-q}^{ij}C_q^{ik} - C_{q-p}^{jk}C_p^{ik}\big) = 0. \tag{2.8}$$

Then (2.3) is equivalent to

$$\big(U_{ij}(s) + U_{jk}(t) - U_{ik}(st)\big)^2 = V_{ij}^{ijk}(s) + V_{jk}^{ijk}(t) - V_{ik}^{ijk}(st) \tag{2.9}$$

with suitable functions V_{ij}^{ijk}, V_{jk}^{ijk} and V_{ik}^{ijk} for $1 \le i < j < k \le n$.

Remark 2.1. If $\big(U_{ij}(t), U_{jk}(t), U_{ik}(t)\big)$ satisfies (2.9) with suitable V_{ij}, V_{jk} and V_{ij}, then $\big(U_{jk}(t), U_{ij}(t), U_{ik}(t)\big)$ and $\big(cU_{ij}(at^r), cU_{jk}(bt^r), cU_{ik}(abt^r)\big)$ have the same property for any complex numbers a, b and c and a positive integer r with $ab \neq 0$.

Proposition 2.2. *The solution (U_{ij}, U_{jk}, U_{ik}) of (2.9) with (2.6) is one of the followings and it satisfies $U_{ijk} = 0$.*

i) *Two of $\{U_{ij}, U_{jk}, U_{ik}\}$ are zero and the other one is any function.*

ii) $(U_{ij}, U_{jk}, U_{ik}) = (at^r, bt^r, ct^{-r})$ *for any a, b and $c \in \mathbb{C}$ and $r \in \mathbb{Z}\setminus\{0\}$.*

iii) $(U_{ij}, U_{jk}, U_{ik}) = \big(\frac{act^r}{1-at^r}, \frac{bct^r}{1-bt^r}, \frac{abct^r}{1-abt^r}\big)$ *with any non-zero complex numbers a, b and c and a positive integer r.*

Proof. All the solutions of the equation (2.9) are obtained by [BP] and [BB] (cf. [OO, Remark 2.3]), which implies this proposition. But we will give a simple proof under the assumption that the origin is at most a pole of U_{ij}, U_{jk} and U_{ik}.

Suppose one of U_{ij}, U_{jk}, U_{ik} is zero and the other two are not zero. If $U_{ij} = 0$ and $C_r^{jk} \neq 0$, then we have $p(m+p)mC_m^{jk}C_p^{ik} = 0$ and therefore $C_p^{ik} = 0$ for $p \neq -r$, $C_{-r}^{ik} \neq 0$ and $C_m^{jk} = 0$ for $m \neq r$. Thus we have ii). We similarly have ii) in the other two cases.

Hence we may assume that any one of $\{U_{ij}, U_{jk}, U_{ik}\}$ is not zero. Define $I_{\ell m} \in \mathbb{Z}\setminus\{0\}$ such that $C_{I_{\ell m}}^{\ell m} \neq 0$ and $C_\nu^{\ell m} = 0$ for $\nu < I_{\ell m}$. Then (2.8) shows

$$I_{ij}I_{jk}(I_{ij} - I_{jk})(C_{I_{ij}}^{ij}C_{I_{jk}}^{jk} - C_{I_{ij}-I_{jk}}^{ij}C_{I_{jk}}^{ik} - C_{I_{jk}-I_{ij}}^{jk}C_{I_{ik}}^{ik}) = 0. \tag{2.10}$$

Suppose $I_{ij} > 0$ and $I_{jk} > 0$. Then (2.10) means $I_{ij} = I_{jk}$, which we put r, and therefore (2.8) with $q = r$ and that with $p = q + r$ mean

$$pr(p-r)(C_p^{ij}C_r^{jk} - C_{p-r}^{ij}C_r^{ik}) = 0 \quad \text{for } p > 0, \tag{2.11}$$

$$(q+r)qr(C_{q+r}^{ij}C_q^{jk} - C_r^{ij}C_q^{ik}) = 0, \tag{2.12}$$

respectively.

If $C_r^{ik} = 0$, it follows from (2.11) that $C_p^{ij} = 0$ for $p \neq r$ by the induction on p and we have similarly $C_q^{jk} = 0$ for $q \neq r$ by the symmetry between U^{ij} and U^{jk} and finally $C_q^{ik} = 0$ for $q \neq -r$ by (2.12). Hence this case is reduced to ii) with $r > 0$.

Suppose $C_r^{ik} \neq 0$. Then by Remark 2.1 we may assume $C_r^{ij} = C_r^{jk} = C_r^{ik}$ by a suitable transformation $(s,t) \mapsto (at, bt)$ and moreover by (2.11) that $U_{ij} = c\sum_{\nu=1}^\infty t^{r\nu}$ and similarly $U_{jk} = c'\sum_{\nu=1}^\infty t^{r\nu}$. Then (2.12) means $U_{ik} = U_{jk} + c''t^{-r}$. Finally we have $c'' = 0$ by (2.10) with $p = 2r$ and $q = -r$ and get $U_{ij} = U_{jk} = U_{ik}$.

Lastly we may assume $I_{ij} < 0$ by Remark 2.1. Then (2.8) with $p = I_{ij} + I_{ik}$ and $q = I_{ik}$ implies $I_{ik} > 0$ and that with $p = I_{ij}$ and $q > 0$ means $C^{jk}_q = 0$ for $q \geq 0$. Hence $I_{jk} < 0$ and similarly we have $C^{ij}_p = 0$ for $p \geq 0$. Moreover (2.8) with $p = q + I_{ij}$ shows $C^{ik}_q = 0$ for sufficiently large integer q. Then $\bigl(U_{ij}(t^{-1}), U_{jk}(t^{-1}), U_{ik}(t^{-1})\bigr)$ is also a solution of (2.9) and this case is reduced to the case when $I_{ij} > 0$ and $I_{jk} > 0$ and therefore we have ii) with $r < 0$.

Note that it is easy to see that the given functions in the proposition satisfy $U_{ijk} = 0$ (cf. Remark 2.1). □

Remark 2.3. If $t = e^{-x}$, then

$$t\frac{d}{dt}\bigl(at^r\bigr) = art^r = are^{-rx},$$
$$t\frac{d}{dt}\Bigl(\frac{at^r}{1-at^r}\Bigr) = \frac{art^r}{(1-at^r)^2} = r\sinh^{-2}\frac{rx - \log a}{2}.$$

3. Type B_2

In this section we study the following commuting differential operators.

$$\begin{cases} P = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + R(x,y), \\ Q = \dfrac{\partial^4}{\partial x^2 \partial y^2} + S \quad \text{ with } \operatorname{ord} S < 4, \\ [P,Q] = 0. \end{cases} \tag{3.1}$$

Note that $P_2 = P^2 - 2Q$ in (1.3). First we review the arguments given in [OO] and [Oc]. Since P is self-adjoint, we may assume Q is also self-adjoint by replacing Q by its self-adjoint part if necessary. Here for $A = \sum a_{ij}(x,y)\frac{\partial^{i+j}}{\partial x^i \partial y^j}$ we define ${}^tA = \sum(-1)^{i+j}\frac{\partial^{i+j}}{\partial x^i \partial y^j}a_{ij}(x,y)$ and A is called self-adjoint if ${}^tA = A$. Then

$$\begin{aligned} R(x,y) &= u^+(x+y) + u^-(x-y) + v(x) + w(y), \\ Q &= \Bigl(\frac{\partial^2}{\partial x \partial y} + \frac{u^+(x+y) - u^-(x-y)}{2}\Bigr)^2 + w(y)\frac{\partial^2}{\partial x^2} + v(x)\frac{\partial^2}{\partial y^2} \\ &\quad + v(x)w(y) + T(x,y), \end{aligned} \tag{3.2}$$

and the function $T(x,y)$ satisfies

(3.3)
$$\begin{aligned} 2\frac{\partial T(x,y)}{\partial x} &= \bigl(u^+(x+y) - u^-(x-y)\bigr)\frac{\partial w(y)}{\partial y} + 2w(y)\frac{\partial}{\partial y}\bigl(u^+(x+y) - u^-(x-y)\bigr), \\ 2\frac{\partial T(x,y)}{\partial y} &= \bigl(u^+(x+y) - u^-(x-y)\bigr)\frac{\partial v(x)}{\partial x} + 2v(x)\frac{\partial}{\partial x}\bigl(u^+(x+y) - u^-(x-y)\bigr). \end{aligned}$$

On the other hand, if a function $T(x,y)$ satisfies (3.3) for suitable functions $u^\pm(t)$, $v(t)$ and $w(t)$, then (3.1) is valid for $R(x,y)$ and Q defined by (3.2).

We have the compatibility condition

(3.4)
$$\begin{aligned} &\frac{\partial}{\partial x}\Bigl(\bigl(u^+(x+y) - u^-(x-y)\bigr)\frac{\partial v(x)}{\partial x} + 2v(x)\frac{\partial}{\partial x}\bigl(u^+(x+y) - u^-(x-y)\bigr)\Bigr) \\ &\quad = \frac{\partial}{\partial y}\Bigl(\bigl(u^+(x+y) - u^-(x-y)\bigr)\frac{\partial w(y)}{\partial y} + 2w(y)\frac{\partial}{\partial y}\bigl(u^+(x+y) - u^-(x-y)\bigr)\Bigr) \end{aligned}$$

for the existence of $T(x,y)$.

Definition 3.1 (Duality in B_2)**.** Under the coordinate transformation

$$(x,y)\mapsto\left(\frac{x+y}{\sqrt{2}},\frac{x-y}{\sqrt{2}}\right) \tag{3.5}$$

the pair $(P,\frac{1}{4}P^2-Q)$ also satisfies (3.1), which we call the *duality* of the commuting differential operators of type B_2.

Denoting $\partial_x=\frac{\partial}{\partial x}$, $\partial_y=\frac{\partial}{\partial y}$ and put

$L=P^2-4Q-(\partial_x^2-\partial_y^2+v(x)-w(y))^2-2u^-(x-y)(\partial_x+\partial_y)^2-2u^+(x+y)(\partial_x-\partial_y)^2.$

Then the order of L is at most 2 and the second order term of L equals

$$\begin{aligned}&2(u^++u^-+v+w)(\partial_x^2+\partial_y^2)-4(u^+-u^-)\partial_x\partial_y-4w\partial_x^2-4v\partial_y^2\\&\qquad-2(v-w)(\partial_x^2-\partial_y^2)-2u^-(\partial_x+\partial_y)^2-2u^+(\partial_x-\partial_y)^2=0.\end{aligned}$$

Since L is self-adjoint, L is of order at most 0 and the 0-th order term of L equals

$$\begin{aligned}&(\partial_x^2+\partial_y^2)(u^++u^-+v+w)+(u^++u^-+v+w)^2-4(vw+T)-2\partial_x\partial_y(u^+-u^-)\\&\qquad-(\partial_x^2-\partial_y^2)(v-w)=(u^++u^-+v+w)^2-4(vw+T)\end{aligned}$$

and therefore we have the following proposition.

Proposition 3.2. i) *By the duality in Definition 3.1 the pair* $\bigl(R(x,y),T(x,y)\bigr)$ *changes into* $\bigl(\tilde{R}(x,y),\tilde{T}(x,y)\bigr)$ *with*

$$\begin{cases}\tilde{R}(x,y)=v\Bigl(\dfrac{x+y}{\sqrt{2}}\Bigr)+w\Bigl(\dfrac{x-y}{\sqrt{2}}\Bigr)+u^+\bigl(\sqrt{2}x\bigr)+u^-\bigl(\sqrt{2}y\bigr),\\ \tilde{T}(x,y)=\dfrac{1}{4}\tilde{R}(x,y)^2-v\Bigl(\dfrac{x+y}{\sqrt{2}}\Bigr)w\Bigl(\dfrac{x-y}{\sqrt{2}}\Bigr)-T\Bigl(\dfrac{x+y}{\sqrt{2}},\dfrac{x-y}{\sqrt{2}}\Bigr).\end{cases} \tag{3.6}$$

ii) *Combining the duality with the scaling map* $R(x,y)\mapsto c^{-2}R(cx,cy)$*, the following pair* $\bigl(R^d(x,y),T^d(x,y)\bigr)$ *defines commuting differential operators if so is* $\bigl(R(x,y),T(x,y)\bigr)$*. This* $R^d(x,y)$ *is also called the dual of* $R(x,y)$*.*

$$\begin{cases}R^d(x,y)=v(x+y)+w(x-y)+u^+(2x)+u^-(2y),\\ T^d(x,y)=\frac{1}{4}R^d(x,y)^2-v(x+y)w(x-y)-T(x+y,x-y).\end{cases} \tag{3.7}$$

Now we give a list of the solutions of (3.4) and (3.3). They are suitable limits of the invariant solutions studied in [OO] and many of them are given in [Oc].

Case I: (Any-A_1)+(Any-A_1) $v=w=0$ and u and v are arbitrary functions.

Case II: $u^+=u^-$, $v=w$ and $(u^+;v)$ is in the following list.

(Trig-B_2) $(\langle\sinh^{-2}\lambda t\rangle;\ \langle\sinh^{-2}2\lambda t,\ \sinh^{-2}\lambda t,\ \cosh 2\lambda t,\ \cosh 4\lambda t\rangle)$,

(Trig-B_2-S) $(\langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda\rangle;\ \langle\sinh^{-2}2\lambda t,\ \cosh 4\lambda t\rangle)$.

Case III: $u^+=u^-$, $(u^+;v,w)$ is in the following list.

(Toda-$D_2^{(1)}$-bry) $(\langle\cosh 2\lambda t\rangle;\ \langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t\rangle,\ \langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t\rangle)$,

(Toda-$D_2^{(1)}$-S-bry) $(\langle\cosh\lambda t,\ \cosh 2\lambda t\rangle;\ \langle\sinh^{-2}\lambda t\rangle,\ \langle\sinh^{-2}\lambda t\rangle)$,

(Toda-$B_2^{(1)}$-bry) $(\langle e^{-2\lambda t}\rangle;\ \langle e^{2\lambda t},\ e^{4\lambda t}\rangle,\ \langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t\rangle)$,

(Toda-$B_2^{(1)}$-S-bry) $(\langle e^{-\lambda t},\ e^{-2\lambda t}\rangle;\ \langle e^{2\lambda t}\rangle,\ \langle\sinh^{-2}\lambda t\rangle)$.

Case IV: $v=w$, $(u^+,u^-;v)$ is in the following list.

(Trig-A_1-bry) $(0,\ \langle\sinh^{-2}\lambda t\rangle;\ \langle e^{-2\lambda t},e^{-4\lambda t},e^{2\lambda t},e^{4\lambda t}\rangle)$,

(Trig-A_1-S-bry) $(0,\ \langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t\rangle;\ \langle e^{-4\lambda t},\ e^{4\lambda t}\rangle)$.

Case V: (u^+, u^-, v, w) is in the following list.

(Toda-$C_2^{(1)}$) $\qquad (0,\ \langle e^{-\lambda t}\rangle,\ \langle e^{\lambda t},\ e^{2\lambda t}\rangle,\ \langle e^{-\lambda t},\ e^{-2\lambda t}\rangle),$

(Toda-$C_2^{(1)}$-S) $\qquad (0,\ \langle e^{-\lambda t},\ e^{-2\lambda t}\rangle,\ \langle e^{2\lambda t}\rangle,\ \langle e^{-2\lambda t}\rangle).$

In the above $\langle\ \rangle$ means an arbitrary linear combination of given functions and, for example, (Trig-B_2) implies

$$\begin{cases} u^+(t) = u^-(t) = C_0 \sinh^{-2} \lambda t, \\ v(t) = w(t) = C_1 \sinh^{-2} 2\lambda t + C_2 \sinh^{-2} \lambda t + C_3 \cosh 2\lambda t + C_4 \cosh 4\lambda t \end{cases}$$

with any complex numbers $C_0, C_1, \ldots, C_4$ and a suitable $\lambda \in \mathbb{C} \setminus \{0\}$.

According to our assumption, put

$$\begin{gathered} t = e^{-y}, \quad s = e^{-x+y}, \\ u^+(x+y) = \sum_{i\ge r} u_i^+ s^i t^{2i},\ u^-(x-y) = \sum_{i\ge r} u_i^- s^i, \\ v(x) = \sum_{j\ge r'} v_j s^j t^j,\ w(y) = \sum_{j\ge r''} w_j t^j, \\ u_i^\pm = v_j = w_k = 0 \quad \text{if } i<r,\ j<r' \text{ and } k<r''. \end{gathered} \tag{3.8}$$

$$\begin{aligned} &\sum_{\substack{i\ge r\\ j\ge r'}} (i+j)(2i+j)v_j(u_i^+ t^{2i+j} - u_i^- t^j)s^{i+j} \\ &= \sum_{\substack{i\ge r\\ j\ge r''}} \Bigl((2i+j)(i+j)w_j u_i^+ t^{2i+j} - (2i-j)(i-j)w_j u_i^- t^j\Bigr)s^i \end{aligned} \tag{3.9}$$

and the coefficients of $s^p t^q$ mean

$$pqv_{2p-q}u_{q-p}^+ - p(2p-q)v_q u_{p-q}^- = q(q-p)w_{q-2p}u_p^+ - (2p-q)(p-q)w_q u_p^-. \tag{3.10}$$

Putting

$$\begin{cases} U^\pm(t) = \sum_{i\ge r} U_i^\pm t^i,\ V(t) = \sum_{j\ge r'} V_j t^j \text{ and } W(t) = \sum_{k\ge r''} W_k t^k, \\ u^\pm(t) = t(U^\pm)'(t) + u_0^\pm,\ v(t) = tV'(t) + v_0 \text{ and } w(t) = tW'(t) + w_0, \end{cases} \tag{3.11}$$

we have

$$pq(2p-q)(p-q)\bigl(V_{2p-q}U_{q-p}^+ + V_q U_{p-q}^- + W_{q-2p}U_p^+ - W_q U_p^-\bigr) = 0, \tag{3.12}$$

which is equivalent to

$$\begin{aligned} V(st)\bigl(U^+(st^2) + U^-(s)\bigr) + W(t)\bigl(U^+(st^2) - U^-(s)\bigr) \\ = F_1(st^2) + F_2(s) + G_1(st) + G_2(t) \end{aligned} \tag{3.13}$$

with suitable functions F_1, F_2, G_1 and G_2 (cf. [Oc, Proposition 2.4]). Thus we have the following proposition.

Proposition 3.3. *For the functions* $(U^\pm, V, W, F_1, F_2, G_1, G_2)$ *satisfying* (3.13) *we have the commuting differential operators* (3.1) *and* (3.2) *by putting*

$$\begin{cases} u^\pm(t) = \partial_t U^\pm(e^t) + C',\ v(t) = \partial_t V(e^t) + C,\ w(t) = \partial_t W(e^t) + C, \\ T(x,y) = \dfrac{1}{2}\Bigl(\partial_x^2 - \partial_y^2\Bigr)\Bigl(V(e^x)\bigl(U^+(e^{x+y}) + U^-(e^{x-y})\bigr) - G_1(e^x)\Bigr) \\ \qquad\qquad + C\bigl(u^+(x+y) + u^-(x-y)\bigr), \\ \qquad C,\ C' \in \mathbb{C}. \end{cases} \tag{3.14}$$

Now we put

$$\mathcal{S}(B_2) = \Big\{\big(U^+(t), U^-(t), V(t), W(t)\big);\ U^\pm, V \text{ and } W \text{ are meromorphic in a neighborhood of 0 and they satisfy (3.13)}\Big\}. \tag{3.15}$$

Remark 3.4. i) Since the constant terms $U_0^\pm$, V_0 and W_0 have no effect on the equation (3.12) and on the original functions $u^\pm$, v and w, we will identify two functions appeared in the solutions of (3.12) if they only differ in their constant terms.

ii) If $\big(U^+(t), U^-(t)\big) = 0$ or $\big(W(t), V(t)\big) = 0$, then (3.12) is always true. We call such $(U^+, U^-, W, V) \in \mathcal{S}(B_2)$ a *trivial solution* of (3.13).

We summarize elementary transformations acting on $\mathcal{S}(B_2)$.

Lemma 3.5. *Let* $\big(U^+(t), U^-(t), V(t), W(t)\big) \in \mathcal{S}(B_2)$.

i) (dual) $\big(V(t), W(t), U^+(t^2), U^-(t^2)\big) \in \mathcal{S}(B_2)$.

ii) (bilinear) *If* $\big(U^+(t), U^-(t), S(t), T(t)\big) \in \mathcal{S}(B_2)$*, then* $\big(aU^+(t), aU^-(t), bV(t) + cS(t), bW(t) + cT(t)\big) \in \mathcal{S}(B_2)$ *for* $a, b, c \in \mathbb{C}$.

iii) (translations) $\big(U^+(ab^2t), U^-(bt), V(abt), W(at)\big) \in \mathcal{S}(B_2)$ *for* $a, b \in \mathbb{C} \setminus \{0\}$.

iv) (scaling) *If* $\big(U^+(t^r), U^-(t^r), W(t^r), V(t^r)\big)$ *is well-defined for a suitable* $r \in \mathbb{Q} \setminus \{0\}$*, it is in* $\mathcal{S}(B_2)$.

v) (symmetry) *If* $W(t)$ *is a rational function, the reflection* $(x, y) \mapsto (x, -y)$ *can be applied to the solution and then* $\big(U^-(t), U^+(t), V(t), -W(t^{-1})\big) \in \mathcal{S}(B_2)$.

vi) (symmetry) *If* $U^-(t)$ *is a rational function, the reflection* $(x, y) \mapsto (y, x)$ *can be applied to the solution and then* $\big(U^+(t), -U^-(t^{-1}), V(t), W(t)\big) \in \mathcal{S}(B_2)$.

The lemma is a direct consequence of the definition of $\mathcal{S}(B_2)$. For example, i) follows from

$$\begin{aligned} U^+(t^2s^2)\big(V(ts^2) + W(t)\big) + U^-(s^2)\big(V(ts^2) - W(t)\big) \\ = F_2(t) + F_1(s^2) + G_2(t^2s^2) + G_1(ts^2). \end{aligned}$$

Note that the transformation in Lemma 3.5 vi) is equals to a certain composition of transformations in Lemma 3.5 i), iv) and v).

Definition 3.6. If a solution of (3.13) obtained by applying transformations in Lemma 3.5 to an original solution, it is called a *standard transformation* of the original solution.

We will study non-trivial solutions of (3.13). Considering standard transformations, we may assume

$$(U_r^+, U_r^-) = (1, 1) \text{ or } (1, 0) \text{ or } (0, 1). \tag{3.16}$$

Proposition 3.7. *Suppose* $\big(U^+(t), U^-(t), V(t), W(t)\big)$ *is a non-trivial solution of* (3.13) *with* (3.11).

i) $U^\pm(t)$*,* $V(t)$ *and* $W(t)$ *are rational functions.*

ii) ([Oc, Theorem 2.3]) *If* $W(t)$ *has a pole at* $t = 1$*, then* $U^+(t) = U^-(t)$ *and* $W(t^{-1}) + W(t) = 0$*. If* $U^-(t)$ *has a pole at* $t = 1$*, then* $V(t) = W(t)$ *and* $U^-(t^{-1}) + U(t) = 0$.

Here we note that this equality is interpreted in the sense of Remark 3.4.

iii) ([Oc, Corollary 3.8]) *If at least two of* $\{U^+(t), U^-(t), V(t), W(t)\}$ *have poles in* $\mathbb{C} \setminus \{0\}$*,* (U^+, U^-, V, W) *is a standard transformation of a solution given in the list* (Trig-B_2) – (Toda-$D_2^{(1)}$-S-bry).

Proof. i) The equation (3.12) shows $W_{q-2r}U_r^+ = W_qU_r^-$ if $q > 2|r| + |r'|$. Hence $W(t)$ is a rational function and therefore so are $U^-(t)$, $U^+(t)$ and $V(t)$ because of Lemma 3.5 i) and v). $\square$

Lemma 3.8. i) *If $V(t)$ has a pole at the origin, then $U^+(t)$ and $U^-(t)$ are holomorphic at the origin.*

ii) *If $U^+(t)$ has a pole at the origin, then $V(t)$ and $W(t)$ are holomorphic at the origin.*

Proof. If $r<0$ and $r'<0$ with $V_{r'}\neq 0$, the coefficients of $s^{r+r'}t^{r'}$ and that of $s^{r+r'}t^{2r+r'}$ in (3.13) show $V_{r'}U_r^-=V_{r'}U_r^+=0$, which contradicts to $(U_r^+,U_r^-)\neq 0$. Thus we have i) and then ii) by Lemma 3.5 i). □

Theorem 3.9. *Any non-trivial solution of* (3.13) *corresponds to a standard transformation of a solution in the list* (Trig-B_2) – (Toda-$C_2^{(1)}$-S).

Proof. We will prove this theorem divided into several cases.

Case 1: One of U^+,U^-,V,W is zero.
Proposition 3.7 assures that we may suppose $V=0$. Then (3.12) turns into

$$pq(2p-q)(p-q)(W_{q-2p}U_p^+-W_qU_p^-)=0. \tag{3.17}$$

Case 1-1: $V=0$, $W_{r''}\neq 0$ and $(U_r^+,U_r^-)=(1,1)$.
Suppose $\bar r:=-r>0$. Then (3.17) with $p=-\bar r$ and $q=r''-2\bar r$ shows

$$\bar r(r''-2\bar r)r''(r''-\bar r)W_{r''}U_r^+=0$$

and hence $r''=\bar r$ or $2\bar r$. Since $\bar r q(2\bar r+q)(\bar r+q)(W_{q+2\bar r}U_r^+-W_qU_r^-)=0$,

$$W(t)=at^{\bar r}(1-t^{\bar r})^{-1}+bt^{2\bar r}(1-t^{2\bar r})^{-1} \tag{3.18}$$

Since $W(t)=at^{\bar r}(1+t^{\bar r})^{-1}$ if $2a+b=0$, we may assume $W(t)$ has a pole at $t=1$ by applying a transformation in Lemma 3.5 iii) and hence $U^+(t)=U^-(t)$ by Proposition 3.7 ii).

On the other hand, (3.17) with $q=r''$ and that with $q=2p+r''$ show

$$\begin{cases} pr''(2p-r'')(p-r'')W_{r''}U_p^-=0 & \text{for } p>0,\\ p(2p+r'')r''(p+r'')W_{r''}U_p^+=0 & \text{for } p<0.\end{cases}$$

Thus we can conclude

$$U^+(t)=U^-(t)=ct^{-\bar r}+dt^{-\frac{\bar r}{2}}+et^{\bar r}+ft^{\frac{\bar r}{2}} \qquad \text{with } bd=bf=0 \tag{3.19}$$

because $\bigl(U^+(t),U^-(t),0,bt^{2\bar r}(1-t^{2\bar r})^{-1}\bigr)\in\mathcal S(B_2)$.

If $r>0$, $rr''(2r-r'')(r-r'')W_{r''}U_r^-=0$ and therefore $r''=2r$ or $r''=r$. Then by putting $\bar r=r$, the equation (3.17) with $p=r$ and $q>r''$ implies (3.18) and hence the same argument as above proves (3.19).

Hence the solution corresponds to a standard transformation of Case IV.

Case 1-2: $V=0$ and $(U_r^+,U_r^-)=(1,0)$ or $(0,1)$.
We have $q(2r-q)(r-q)W_q=0$ or $pq(2r-q)(r-q)W_{q-2r}=0$. Hence $W(t)=at^{\bar r}+bt^{2\bar r}$ with $b\neq 0$ and $\bar r\in\mathbb Z\setminus\{0\}$

If $a=0$, then (3.17) with $q=2\bar r$ implies $U_p^-=0$ for $p\neq 0$, $\bar r$, $2\bar r$. If $a\neq 0$, then (3.17) with $(p,q)=(\frac{\bar r}{2},3\bar r)$ implies $U^+_{\frac{\bar r}{2}}=0$, that with $q=2\bar r$ implies $U_p^-=0$ for $p\neq 0$, $\bar r$, $2\bar r$ and that with $q=\bar r$ implies $U_p^-=0$ for $p\neq 0$, $\pm\frac{\bar r}{2}$, $\bar r$. Hence $U^-(t)=c^-t^{\bar r}+d^-t^{2\bar r}$ with $ad^-=0$.

Since $\bigl(U^+(t^{-1}),U^-(t^{-1}),0,W(t^{-1})\bigr)\in\mathcal S(B_2)$, we have $U^+(t)=c^+t^{-\bar r}+d^+t^{-2\bar r}$ with $ad^+=0$ and the solution corresponds to the standard transform of Case V.

Now we may assume that none of $U^\pm(t),V(t),W(t)$ is zero and

$$V_{r'}\neq 0 \quad\text{and}\quad W_{r''}\neq 0. \tag{3.20}$$

Lemma 3.8 and Lemma 3.5 i) assure that we may assume $r>0$ except for the following case.

Case 2: W and U^- have poles and V and U^+ are holomorphic at the origin. Then $r'' < 0$ and $r < 0$. Put $\bar{r} = -r$. The coefficients of $s^{-\bar{r}}t^q$ in (3.13) imply $W(t) = at^{-2\bar{r}} + bt^{-\bar{r}}$.

Case 2-1: $W(t) = t^{-2\bar{r}} + bt^{-\bar{r}}$.
The coefficients of $s^p t^{-2\bar{r}}$ imply $U^-(s) = s^{-\bar{r}}$. Note that $U^-(t^{-1})$ and $W(t^{-1})$ are holomorphic at the origin and $\big(U^+(t^{-1}), U^-(t^{-1}), V(t^{-1}), W(t^{-1})\big) \in \mathcal{S}(B_2)$. If $U^+(t^{-1})$ has a pole at the origin, Lemma 3.8 assures that $V(t^{-1})$ is holomorphic there. Hence we may assume $r > 0$ by using a transformation in Lemma 3.5 i) if necessary.

Case 2-2: $W(t) = t^{-\bar{r}}$.
The coefficients $s^p t^{-\bar{r}}$ in (3.13) imply $U^-(s) = s^{-\bar{r}} + cs^{-\frac{\bar{r}}{2}}$ and this case is reduced to the previous case by Lemma 3.5 i).

Now we may assume

$$r > 0. \tag{3.21}$$

Case 3: $r > 0$ and $r' > 0$.
Putting $p = r$ in (3.12), we have

$$q(2r - q)(r - q)(W_{q-2r}U_r^+ - W_q U_r^-) = 0. \tag{3.22}$$

Case 3-1: $(U_r^+, U_r^-) = (1, 0)$ or $(0, 1)$.
Owing to Lemma 3.5 v), we may assume $U_r^- = 0$. Then (3.22) with $q = r'' + 2r$ means $r'' = -2r$ or $r'' = -r$ and (3.12) with $q = r''$ means $U^- = 0$. Hence this case is reduced to Case 1.

Case 3-2: $(U_r^+, U_r^-) = (1, 1)$.
The equation (3.22) with $q = r''$ means $r'' = r$ or $r'' = 2r$.

Note that (3.22) means $W(t) = at^r(1 - t^r)^{-1} + bt^{2r}(1 - t^{2r})^{-1}$. Since $U^\pm$, V and W are holomorphic at the origin, Lemma 3.5 i) assures that if $r' \neq r''$, this case is reduced to Case 3-1. Hence we may assume $r' = r''$ and therefore $(V_{r'}, W_{r'}) = (1, 1)$ by a suitable translation $s \mapsto as$. It also follows from Lemma 3.5 i) that $U^-(s) = cs^{\frac{r'}{2}}(1 - s^{\frac{r'}{2}})^{-1} + ds^{r'}(1 - s^{r'})^{-1}$ and then this case is reduced to Proposition 3.7 iii).

Case 4: $r > 0$ and $r' < 0$.
Using the transformation in Lemma 3.5 v) if necessary, we may assume

$$(U_r^+, U_r^-) = (\delta, 1) \tag{3.23}$$

with $\delta = 0$ or 1. The equation (3.12) with $p = r + m$, $q - p = \pm r$ and $m < 0$ means $(r + m)(2r + m)U_r^\pm V_m = 0$ and therefore

$$\begin{aligned} &V(t) = at^{-2r} + bt^{-r} + \sum_{j>0} V_j t^j, \\ &r' = \begin{cases} -2r & \text{if } a \neq 0, \\ -r & \text{if } a = 0. \end{cases} \end{aligned} \tag{3.24}$$

Here $(a, b) \neq (0, 0)$. Moreover (3.12) with $p = r$ shows

$$U_r^+ W_{q-2r} = U_r^- W_q \quad \text{if } q \notin \{-2r, -r, 0, r, 2r, 3r, 4r\}. \tag{3.25}$$

Put $\bar{r} = -\max\{r', r''\} > 0$. Then (3.12) with $q = -\bar{r}$ means

$$p\bar{r}(2p + \bar{r})(p + \bar{r})(V_{-\bar{r}}U_{p+\bar{r}}^- - W_{-\bar{r}}U_p^-) = 0. \tag{3.26}$$

Similarly (3.12) with $q = -r$ means

$$p\bar{r}(2p + r)(p + r)(bU_{p+r}^- - W_{-r}U_p^-) = 0. \tag{3.27}$$

If $r' > r''$, we have $U^- = 0$ because $V_{-\bar{r}} = 0$. Hence $r' \le r''$ and we may moreover assume

$$(V_{-\bar{r}}, W_{-\bar{r}}) = (1, \epsilon) \quad \text{with } \epsilon = 0 \text{ or } 1 \quad \text{and} \quad \bar{r} = \begin{cases} 2r & \text{if } a \neq 0, \\ r & \text{if } a = 0. \end{cases} \tag{3.28}$$

Then (3.25) implies

$$W(t) = e_1 t^r (1-\delta t^r)^{-1} + e_2 t^{2r}(1-\delta t^{2r})^{-1} + e_3 t^{-2r} + e_4 t^{-r} + e_5 t^r + e_6 t^{2r} + e_7 t^{3r} + e_8 t^{4r}$$

and it follows from (3.23), (3.26) and (3.28) that

$$U^-(s) = s^r(1-\epsilon s^r)^{-1} + cs^{2r}(1-\epsilon s^{2r})^{-1}. \tag{3.29}$$

If $\epsilon = 1$, we may assume that $U^-(s)$ has a pole at $s = 1$ as in the argument in Case 1-1 and then $b = W_r$ in (3.27). Note that if $b \neq 0$, (3.27) implies $c = 0$. Hence

$$bc = 0 \tag{3.30}$$

and (3.27) with $p = r$ means

$$e_4 = 0 \quad \text{if } \epsilon = 0. \tag{3.31}$$

Case 4-1: $(e_1, e_2) \neq 0$ and $\delta = 1$.
We may also assume $W(t)$ has a pole at $t = 1$. If $\epsilon = 1$, then $U^-(t)$ and $W(t)$ have poles in $\mathbb{C} \setminus \{0\}$ and this case is reduced to Proposition 3.7 iii). Hence we may assume $\epsilon = 0$ and therefore $e_3 = 0$ by (3.28). Then Proposition 3.7 ii) assures

$$\begin{aligned} W(t) &= e_1 t^r(1-t^r)^{-1} + e_2 t^{2r}(1-t^{2r})^{-1}, \\ U^+(s) &= U^-(s) = s^r + cs^{2r}. \end{aligned} \tag{3.32}$$

Now (3.12) with $(p, q) = (2r, r)$ means $ce_1 = 0$. Thus $(U^+(t), U^-(t), V(t), 0) \in \mathcal{S}(B_2)$ and it follows from Case 1 that $V(t) = at^{-2r} + bt^{-r}$ with $bc = 0$. Then the solution corresponds to (Toda-$B_2^{(1)}$-bry) or (Toda-$B_2^{(1)}$-S-bry).

Case 4-2. $e_1 = e_2 = 0$, $\epsilon = 1$.
Proposition 3.7 ii) assures $V(t) = W(t) = at^{-2r} + bt^{-r} + e_5 t^r + e_6 t^{2r} + e_7 t^{3r} + e_8 t^{4r}$ with $bc = 0$. Putting $q = 2p - \bar{r}$ in (3.12) we have $V_{\bar{r}} U^+_{p-\bar{r}} + W_{-\bar{r}} U^+_p = 0$ if p is sufficiently large positive integer. Hence if $U^+(t)$ is not a polynomial of t, it has a pole in $\mathbb{C} \setminus \{0\}$ and this case is reduced to Proposition 3.7 iii).

Thus we may assume $U^+(s) = \sum_{i=r}^N U_i^+ s^i$ with $U_N^+ \neq 0$. Suppose $W_j = 0$ for $j > M$. If $M > 0$, the coefficients of $s^N t^{M+2N}$ in (3.13) implies $W_M U_N^+ = 0$. Hence $e_5 = e_6 = e_7 = e_8 = 0$ and therefore $(U^+, 0, V, W) \in \mathcal{S}(B_2)$ and Case 1 implies $U^+(s) = s^r + c' s^{2r}$ with $bc' = 0$. The solution is a standard transform of case V.

Case 4-3: $e_1 = e_2 = \epsilon = 0$.
Then $U^-(s) = s^r + cs^{2r}$ and $W(t) = e_5 t^r + e_6 t^{2r} + e_7 t^{3r} + e_8 t^{4r}$. Putting $p = q + r$ in (3.12), we have $V_q U_r^- - W_q U_{q+r}^- = 0$ for $q > 0$ and therefore $V(t) = at^{-2r} + bt^{-r} + ce_5 t^r$.

Suppose $U^+(s)$ is not a polynomial of s. Putting $q = 2p + \bar{r}$ in (3.12), we have $U^+_{p+\bar{r}} + W_{\bar{r}} U^+_p = 0$ for a sufficiently large p. We may assume $U^+(s)$ has a pole at $s = 1$. Then $W_{\bar{r}} = -1$ and Proposition 3.7 ii) with Lemma 3.5 proves $V(t^{-1}) + W(t) = 0$ and $V(t) = at^{-2r} + bt^{-r}$ with $bc = 0$. Thus $(U^+, 0, V, W) \in \mathcal{S}(B_2)$ and $U^+(s) = d_1 s^r (1-s^r)^{-1} + d_2 s^{2r}(1-s^{2r})^{-1}$ with $bd_2 = 0$ and the solution is a standard transform of Case IV.

Now we may assume $U^+(s) = \sum_{i=r}^N U_i^+ s^i$ and $W(t) = \sum_{i=r}^M W_i t^i$ with $W_M U_N^+ \neq 0$. Then (3.12) with $(p, q) = (N, M + 2N)$ shows $W_M U_N^+ = 0$, which contradicts to the assumption. $\square$

Corollary 3.10. *The non-trivial solutions $R(x,y)$ of* (3.1) *with regular singularity at the point $t=0$ are transformations of the following solutions under translations.*

(Trig-BC_2-reg)

$$\begin{aligned}&C_1\big(\sinh^{-2}\lambda(x+y)+\sinh^{-2}\lambda(x-y)\big)+C_2\big(\sinh^{-2}\lambda x+\sinh^{-2}\lambda y\big)\\&\quad+C_3\big(\sinh^{-2}2\lambda x+\sinh^{-2}2\lambda y\big)+C_0,\end{aligned}$$

(Trig^d-BC_2-reg)

$$\begin{aligned}&C_1\big(\sinh^{-2}\lambda(x+y)+\sinh^{-2}\lambda(x-y)\big)\\&\quad+C_2\big(\sinh^{-2}2\lambda(x+y)+\sinh^{-2}2\lambda(x-y)\big)\\&\quad+C_3\big(\sinh^{-2}2\lambda x+\sinh^{-2}2\lambda y\big)+C_0,\end{aligned}$$

(Toda-D_2-bry)

$$C_1\big(e^{-2\lambda(x+y)}+e^{-2\lambda(x-y)}\big)+C_2\sinh^{-2}\lambda y+C_3\sinh^{-2}2\lambda y+C_0,$$

(Toda^d-D_2-bry)

$$C_1\sinh^{-2}\lambda(x-y)+C_2\sinh^{-2}2\lambda(x-y)+C_3\big(e^{-4\lambda x}+e^{-4\lambda y}\big)+C_0,$$

(Trig-A_1-bry-reg)

$$C_1\sinh^{-2}\lambda(x-y)+C_2\big(e^{-2\lambda x}+e^{-2\lambda y}\big)+C_3\big(e^{-4\lambda x}+e^{-4\lambda y}\big)+C_0,$$

(Trig^d-A_1-bry-reg)

$$\begin{aligned}&C_1\big(e^{-\lambda(x+y)}+e^{-\lambda(x-y)}\big)+C_2\big(e^{-2\lambda(x+y)}+e^{-2\lambda(x-y)}\big)\\&\quad+C_3\sinh^{-2}\lambda y+C_0,\end{aligned}$$

(Toda-BC_2)

$$C_1e^{-\lambda(x-y)}+C_2e^{-\lambda y}+C_3e^{-2\lambda y}+C_0,$$

(Toda^d-BC_2)

$$C_1e^{-\lambda(x-y)}+C_2e^{-2\lambda(x-y)}+C_3e^{-2\lambda y}+C_0.$$

4. Type B_n $(n\geq 3)$

Let $\mathbb{R}^n$ be the Euclidean space with the natural inner product $\langle x,y\rangle=\sum_{i=1}^n x_iy_i$ for $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n$. Then $e_i=(\delta_{i1},\dots,\delta_{i\nu},\dots,\delta_{in})$ for $i=1,\dots,n$ form a natural orthonormal basis of $\mathbb{R}^n$. For $v\in\mathbb{R}^n$, let ∂_v be the differential operator defined by $(\partial_v\psi)(x)=\dfrac{d\psi(x+tv)}{dt}\Big|_{t=0}$ for a function $\psi(x)$ on $\mathbb{R}^n$ and we put $\partial_i=\partial_{e_i}$. If $v\neq 0$, the reflection w_v with respect to v is a linear transformation of $\mathbb{R}^n$ defined by $w_v(x)=x-\dfrac{2\langle v,x\rangle}{\langle v,v\rangle}v$ for $x\in\mathbb{R}^n$.

The root system $\Sigma=\Sigma(B_n)$ of type B_n is realized in $\mathbb{R}^n$ by

$$\begin{cases}\Sigma(A_{n-1})^+=\{e_i-e_j;\ 1\leq i<j\leq n\},\\ \Sigma(D_n)^+=\Sigma_L^+=\{e_i\pm e_j;\ 1\leq i<j\leq n\},\\ \Sigma(D_n)=\Sigma_L=\{\alpha,-\alpha;\ \alpha\in\Sigma(D_n)^+\},\\ \Sigma(B_n)_S^+=\Sigma_S^+=\{e_k;\ 1\leq k\leq n\},\\ \Sigma(B_n)^+=\Sigma^+=\Sigma(D_n)^+\cup\Sigma(B_n)_S^+,\\ \Sigma(B_n)=\{\alpha,\ -\alpha;\ \alpha\in\Sigma(B_n)^+\}.\end{cases}\tag{4.1}$$

The Weyl group W_Σ of Σ is the finite group generated by $\{w_\alpha;\ \alpha\in\Sigma\}$, which is the group generated by the permutation of the coordinate $(x_1,\dots,x_n)$ of $\mathbb{R}^n$ and by the change of the signs of some coordinates x_i. For a subset F of Σ, let W_F denote the subgroup of W_Σ generated by $\{w_\alpha;\ \alpha\in F\}$. Then we call the set

$\bar{F} = \{w\alpha;\ w \in W_F \text{ and } \alpha \in F\}$ the root system generated by F and W_F the Weyl group of the root system $\bar{F}$. Let

$$P = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + R(x) \tag{4.2}$$

be a differential operator with a function $R(x)$ such that it admits a differential operator

$$Q = \sum_{j=1}^{n} \frac{\partial^4}{\partial x_j^4} + S \qquad \text{with } \operatorname{ord} S < 4 \tag{4.3}$$

satisfying $PQ = QP$.

Now we assume

$$\begin{aligned} R(x) &= \sum_{\alpha \in \Sigma(B_n)^+} u_\alpha(\langle \alpha, x\rangle) \\ &= \sum_{1 \le i < j \le n} \big(u^{ij}(x_i + x_j) + v^{ij}(x_i - x_j)\big) + \sum_{k=1}^{n} w^k(x_k), \\ u^{ij} &= u_{e_i - e_j},\ v^{ij} = u_{e_i + e_j} \quad \text{and} \quad w^k = u_{e_k} \\ &\quad \text{for } 1 \le i < j \le n \text{ and } 1 \le k \le n. \end{aligned} \tag{4.4}$$

For $\alpha \in \Sigma(B_n)^+$, we put $u_{-\alpha}(t) = u_\alpha(-t)$ for the convention.

Fix indices i and j with $1 \le i < j \le n$ and put $u^{ji}(t) = u^{ij}(-t)$ and $I(i,j) = \{1, \dots, n\} \backslash \{i, j\}$. It follows from the proof of [OOS, Theorem 6.1] that the condition for the existence of Q is equivalent to

$$S_{ij} = S_{ji} \qquad (1 \le i < j \le n) \tag{4.5}$$

with

$$\begin{aligned} S^{ij} = {} & \Big(\partial_i^2 w^i(x_i) + \sum_{\nu \in I(i,j)} \partial_i^2\big(u^{i\nu}(x_i + x_\nu) + v^{i\nu}(x_i - x_\nu)\big)\Big) \\ & \quad \cdot \Big(u^{ij}(x_i + x_j) - v^{ij}(x_i - x_j)\Big) \\ & + 3\Big(\partial_i w^i(x_i) + \sum_{\nu \in I(i,j)} \partial_i\big(u^{i\nu}(x_i + x_\nu) + v^{i\nu}(x_i - x_\nu)\big)\Big) \\ & \quad \cdot \Big(\partial_i u^{ij}(x_i + x_j) - \partial_i v^{ij}(x_i - x_j)\Big) \\ & + 2\Big(w^i(x_i) + \sum_{\nu \in I(i,j)} \big(u^{i\nu}(x_i + x_\nu) + v^{i\nu}(x_i - x_\nu)\big)\Big) \\ & \quad \cdot \Big(\partial_i^2 u^{ij}(x_i + x_j) - \partial_i^2 v^{ij}(x_i - x_j)\Big) \\ & + \sum_{\nu \in I(i,j)} \Big(\partial_i^2 u^{i\nu}(x_i + x_\nu) - \partial_i^2 v^{i\nu}(x_i - x_\nu)\Big)\Big(u^{j\nu}(x_j + x_\nu) - v^{j\nu}(x_j - x_\nu)\Big). \end{aligned}$$

Then we have assumed that

$$u_\alpha(\log t) = \sum u_\nu^\alpha t^\nu \quad \text{for } \alpha \in \Sigma^+ \tag{4.6}$$

with $u_\nu^\alpha \in \mathbb{C}$. Here $u_\alpha(\log t)$ is analytic if $0 < |t| \ll 1$ and $u_\nu^\alpha = 0$ if ν is a sufficiently large negative integer.

Put

$$(4.7)\quad \begin{aligned} t_j &= e^{-x_j+x_{j+1}}\ (j=1,\dots,n-1),\quad t_n = e^{-x_n},\\ u^{ij}(x_i+x_j) &= \sum u^{ij}_\nu t_i^\nu\cdots t_{j-1}^\nu t_j^{2\nu}\cdots t_n^{2\nu},\\ v^{ij}(x_i-x_j) &= \sum v^{ij}_\nu t_i^\nu\cdots t_{j-1}^\nu,\\ w^i(x_k) &= \sum w^i_\nu t_k^\nu\cdots t_n^\nu,\\ U_\alpha(t) &= \sum_{\nu\in\mathbb{Z}\setminus\{0\}} U^\alpha_\nu t^\nu \quad\text{with } u^\alpha_\nu = \nu U^\alpha_\nu \text{ and } \alpha\in\Sigma^+. \end{aligned}$$

Here $1\le i<j\le n$ and u^{ij}_ν, v^{ij}_ν and $w^i_\nu\in\mathbb{C}$ and they are zero if ν is a sufficiently big negative integer. Then the coefficients of $(t_i\cdots t_{j-1})^q(t_j\cdots t_n)^p$ in (4.5) show

$$pqw^i_{2p-q}u^{ij}_{q-p} - p(2p-q)w^j_q v^{ij}_{p-q} = q(q-p)w^i_{q-2p}u^{ij}_p - (2p-q)(p-q)w^j u^{ij}_p$$

if $pq(p-q)(2p-q)\ne 0$ and therefore by putting

$$(4.8)\quad \begin{cases} U^\pm(t) = \sum_{\nu\ge r} U^\pm_i t^\nu,\ V(t) = \sum_{\nu\ge r} V_\nu t^\nu \text{ and } W(t) = \sum_{\nu\ge r''} W_\nu t^\nu,\\ U^\pm_0 = V_0 = W_0 = 0,\\ u^{ij}(t) = t(U^+)'(t) + u^{ij}_0,\ v^{ij}(t) = t(V^-)'(t) + v^{ij}_0,\\ w^i(t) = tV'(t) + w^i_0 \text{ and } w^j(t) = tW'(t) + w^j_0, \end{cases}$$

we have (3.12) and (3.13). Hence $(u^{ij}, v^{ij}, w^i, w^j)$ is a standard transformation of a solution of type B_2 studied in §3.

Suppose $\{\alpha,\beta,\alpha+\beta\}\subset\Sigma^+_L$. Then $(\alpha,\beta,\alpha+\beta)$ is one of the followings

$$(4.9)\quad (e_{i_1-i_2}, e_{i_2-i_3}, e_{i_1-i_3}),$$

$$(4.10)\quad (e_{i_1-i_2}, e_{i_2+i_3}, e_{i_1+i_3}),$$

$$(4.11)\quad (e_{i_2-i_3}, e_{i_1+i_3}, e_{i_1+i_2}),$$

$$(4.12)\quad (e_{i_1-i_3}, e_{i_2+i_3}, e_{i_1+i_2})$$

with $1\le i_1<i_2<i_3\le n$.

Put $s=e^{-\langle\alpha,x\rangle}$ and $t=e^{-\langle\beta,x\rangle}$. Moreover put $(i,j,\nu)=(i_1,i_2,i_3)$, (i_1,i_2,i_3), (i_2,i_3,i_1) and (i_1,i_3,i_2) according to (4.9), (4.10), (4.11) and (4.12), respectively. Then $(u_\alpha,u_\beta,u_{\alpha+\beta}) = (v^{ij},v^{j\nu},v^{i\nu})$, $(v^{ij},u^{j\nu},u^{i\nu})$, $(v^{ij},u^{j\nu},u^{i\nu})$ and $(v^{ij},u^{j\nu},u^{i\nu})$, respectively, and the coefficients of s^pt^q in (4.5) show

$$(4.13)\quad \begin{aligned} &(-q^2-3(p-q)q-2(p-q)^2)u^{\alpha+\beta}_q u^\alpha_{p-q} + p^2u^{\alpha+\beta}_p u^\beta_{q-p}\\ &\qquad= (-q^2+3pq-2p^2)u^\beta_q u^\alpha_p + (q-p)^2u^{\alpha+\beta}_p u^\beta_{q-p}. \end{aligned}$$

if $pq(p-q)\ne 0$. Hence

$$(4.14)\quad pq(p-q)(2p-q)(U^\alpha_p U^\beta_q - U^\alpha_{p-q}U^{\alpha+\beta}_q - U^\beta_{q-p}U^{\alpha+\beta}_p) = 0.$$

Now put $(i,j,\nu) = (i_2,i_3,i_1)$, (i_1,i_3,i_1), (i_1,i_3,i_2) and (i_2,i_3,i_1) according to (4.9), (4.10), (4.11) and (4.12), respectively. Then $(u_\alpha,u_\beta,u_{\alpha+\beta}) = (v^{i\nu},v^{ij},v^{j\nu})$, $(v^{i\nu},u^{ij},u^{j\nu})$, $(v^{j\nu},u^{ij},u^{i\nu})$ and $(v^{j\nu},u^{ij},u^{i\nu})$, respectively, and the coefficients of s^pt^q in (4.5) show

$$(4.15)\quad \begin{aligned} &(p^2+3(q-p)p+2(q-p)^2)u^{\alpha+\beta}_p u^\beta_{q-p} - q^2u^{\alpha+\beta}_q u^\alpha_{p-q}\\ &\qquad= (p^2-3pq+2q^2)u^\alpha_p u^\beta_q - (p-q)^2u^{\alpha+\beta}_q u^\alpha_{p-q} \end{aligned}$$

if $pq(p-q)\ne 0$ and we have

$$(4.16)\quad pq(p-q)(p-2q)(U^\alpha_p U^\beta_q - U^\alpha_{p-q}U^{\alpha+\beta}_q - U^\beta_{q-p}U^{\alpha+\beta}_p) = 0.$$

Combining (4.14) and (4.16), we get

$$pq(p-q)(U_p^\alpha U_q^\beta - U_{p-q}^\alpha U_q^{\alpha+\beta} - U_{q-p}^\beta U_p^{\alpha+\beta}) = 0. \tag{4.17}$$

Namely,

$$\big(U_\alpha(s) + U_\beta(t) - U_{\alpha+\beta}(st)\big)^2 = F_\alpha(s) + F_\beta(t) + F_{\alpha+\beta}(st) \tag{4.18}$$

with suitable functions F_α, F_β and $F_{\alpha+\beta}$. Then if at least two in $\{U_\alpha, U_\beta, U_{\alpha+\beta}\}$ do not vanish, Proposition 2.2 shows that $\big(U_\alpha(t), U_\beta(t), U_{\alpha+\beta}(t)\big)$ is a standard transformation of $\big(t^r(1-t^r)^{-1}, t^r(1-t^r)^{-1}, t^r(1-t^r)^{-1}\big)$ or $(C_1t^r, C_2t^r, C_3t^{-r})$.

The argument above shows the following lemma.

Lemma 4.1. *Let α and $\beta \in \Sigma$ such that $\alpha \neq \pm\beta$, $\langle\alpha,\beta\rangle \neq 0$, $|\alpha| \geq |\beta|$, $U_\alpha \neq 0$ and $U_\beta \neq 0$. Suppose*

$$U_\alpha(t) = C_1 t^r.$$

Then we have the following two cases.

Case 1: $|\alpha| = |\beta|$.

$$U_\beta(t) = \begin{cases} C_1' t^r & \text{if } \langle\alpha,\beta\rangle < 0, \\ C_1' t^{-r} & \text{if } \langle\alpha,\beta\rangle > 0. \end{cases}$$

Case 2: $|\alpha|^2 = 2|\beta|^2$.

$U_\beta(t) = C_1' t^r(1-t^r) + C_2' t^{2r}(1-t^{2r})$ *and* $U_{w_\beta(\alpha)}(t) = U_\alpha(t)$ *under a translation or*

$$U_\beta(t) = \begin{cases} C_1' t^r + C_2' t^{2r} & \text{if } \langle\alpha,\beta\rangle < 0, \\ C_1' t^{-r} + C_2' t^{-2r} & \text{if } \langle\alpha,\beta\rangle > 0. \end{cases}$$

Definition 4.2. For the functions u_α in (4.4), put

$$\Delta = \{\alpha \in \Sigma(B_n)^+;\ u_\alpha' \neq 0\}. \tag{4.19}$$

Let $\bar\Delta$ be the root system generated by Δ and let

$$\bar\Delta = \bar\Delta_1 \cup \cdots \cup \bar\Delta_N \tag{4.20}$$

be the decomposition into irreducible root systems. Put

$$\Delta_k = \bar\Delta_k \cap \Delta \tag{4.21}$$

and we call it an *irreducible component* of Δ.

We say that P with the potential function $R(x)$ is *irreducible* if $\bar\Delta$ is an irreducible root system of rank n or of type A_{n-1}.

Remark 4.3. i) $\Delta = \Delta_1 \cup \cdots \cup \Delta_N$.

ii) Suppose $\alpha \in \Delta_k$. Then $\beta \in \Delta$ is an element of Δ_k if and only if there exists a sequence $\alpha_1 = \alpha,\ \alpha_2, \ldots, \alpha_\ell = \beta \in \Delta$ such that $\langle\alpha_i, \alpha_{i+1}\rangle \neq 0$ for $i = 1, \ldots, \ell-1$.

Lemma 4.4. *Let Δ' be an irreducible component of Δ and let $\bar\Delta'$ and $\bar\Delta_L'$ be the root systems generated by Δ' and $\Delta_L' := \Delta' \cap \Sigma_L$, respectively. Suppose $\bar\Delta'$ is of type B_m with $m > 2$. Then $\bar\Delta_L'$ is of type A_{m-1} or type D_m.*

Proof. Since Δ_L' and $\{e_1, \ldots, e_m\}$ generate Δ', there exist $\alpha_i \in \Delta_L'$ such that $\langle\alpha_i, e_i\rangle \neq 0$ and $\langle\alpha_i, e_{i+1}\rangle \neq 0$ for $1 \leq i < m$. Since $\alpha_1, \ldots, \alpha_{m-1}$ generate a root system of type A_{m-1}, $\bar\Delta_L'$ is of type A_{m-1} or type D_m. $\square$

Lemma 4.5. *Let S be a subset of a classical root system $\tilde\Sigma$ of type A or B. Suppose S generates an irreducible root system $\bar S$ and*

$$\langle\alpha,\beta\rangle \leq 0 \quad \text{for } \alpha \in S \text{ and } \beta \in S \text{ with } \alpha \neq \beta.$$

If the rank of $\bar{S}$ is larger than one, S is the image of one of the following sets under the suitable transformation by an element of the Weyl group of $\tilde{\Sigma}$.

$$\{e_1 - e_2, \ldots, e_{m-1} - e_m\} \qquad (m \geq 3), \tag{4.22}$$

$$\{e_1 - e_2, \ldots, e_{m-1} - e_m, e_m\} \qquad (m \geq 2), \tag{4.23}$$

$$\{e_1 - e_2, \ldots, e_{m-1} - e_m,\, e_{m-1} + e_m\} \qquad (m \geq 4), \tag{4.24}$$

$$\{e_1 - e_2, \ldots, e_{m-1} - e_m,\, e_m - e_1\} \qquad (m \geq 3), \tag{4.25}$$

$$\{e_1 - e_2, \ldots, e_{m-1} - e_m,\, e_m,\, -e_1\} \qquad (m \geq 2), \tag{4.26}$$

$$\{e_1 - e_2, \ldots, e_{m-1} - e_m,\, e_{m-1} + e_m,\, -e_1 - e_2\} \qquad (m \geq 4), \tag{4.27}$$

$$\{e_1 - e_2, \ldots, e_{m-1} - e_m,\, e_{m-1} + e_m,\, -e_1\} \qquad (m \geq 3). \tag{4.28}$$

Proof. Let S' be a subset of S such that S' is a transformation of one of the sets $\{e_1\}$, $\{e_1 - e_2\}$, (4.22), (4.23) and (4.24) by an element of the Weyl group of $\tilde{\Sigma}$.

If the number of the elements of S' is smaller than the rank of $\bar{S}$, there exists $\alpha \in S$ and $\beta \in S'$ such that $\langle \alpha, \beta \rangle \neq 0$ and $\alpha \notin \sum_{\gamma \in S'} \mathbb{R}\gamma$. Then it is easy to see that $S' \cup \{\alpha\}$ is a transformation of (4.22), (4.23) or (4.24) by a suitable element of the Weyl group.

Thus we may assume that S' equals (4.22) or (4.23) or (4.24) and that the number of the elements of S' equals the rank of $\bar{S}$. Put $S_o = \{\beta \in \tilde{\Sigma} \cap \sum_{\gamma \in S'} \mathbb{R}\gamma; \langle \beta, \gamma \rangle \leq 0$ for any $\gamma \in S'\}$. Then if S' is (4.22) or (4.23), $S_o = \{e_m - e_1\}$ or $\{-e_1, -e_1 - e_2\}$, respectively. If S' is (4.24), then $S_o = \{-e_1 - e_2, -e_1\}$ or $\{-e_1 - e_2\}$ according to $\tilde{\Sigma}$ is of type B or A, respectively. Thus the lemma is clear because $S' \subset S \subset S' \cup S_o$. □

Lemma 4.6. *Let $\bar{\Delta}$ be a root system generated by a classical root system $\tilde{\Sigma}$ of type A_{n-1} or B_n. Let $\bar{\Delta} = \bar{\Delta}_1 \cup \cdots \cup \bar{\Delta}_{N'} \cup \cdots \cup \bar{\Delta}_N$ be an irreducible decomposition so that the rank of $\bar{\Delta}_i$ is larger than one for $1 \leq i \leq N'$ and the rank of $\bar{\Delta}_i$ equals one for $N' < i \leq N$. Then under the transformation by an element of the Weyl group of Σ, there exists a sequence of integers $0 = n_0 < n_1 < n_2 < \cdots < n_{N'}$ such that $\bar{\Delta}_i$ is generated by*

$$\begin{cases} \{e_{n_{i-1}+1} - e_{n_{i-1}+2}, \ldots, e_{n_i-1} - e_{n_i}\} & \text{if } \bar{\Delta}_i \text{ is of type } A, \\ \{e_{n_{i-1}+1} - e_{n_{i-1}+2}, \ldots, e_{n_i-1} - e_{n_i},\, e_{n-i-1} + e_{n_i}\} & \text{if } \bar{\Delta}_i \text{ is of type } D, \\ \{e_{n_{i-1}+1} - e_{n_{i-1}+2}, \ldots, e_{n_i-1} - e_{n_i},\, e_{n_i}\} & \text{if } \bar{\Delta}_i \text{ is of type } B \end{cases}$$

for $1 \leq i \leq N'$. Moreover if $N' < i \leq N$. $\bar{\Delta}_i$ equals $\{\pm e_\nu\}$, $\{\pm(e_\nu - e_{\nu+1})\}$ or $\{\pm(e_\nu + e_{\nu+1})\}$ for a suitable ν with $\nu > n_{N'}$.

Proof. Note that $\{\alpha \in \tilde{\Sigma};\, \langle e_1 - e_2, \alpha \rangle = 0\}$ is generated by

$$\begin{cases} \{e_3 - e_4, \ldots, e_{n-1} - e_n\} & \text{if } \tilde{\Sigma} \text{ is of type } A_n, \\ \{e_3 - e_4, \ldots, e_{n-1} - e_n,\, e_n\} \text{ and } \{e_1 + e_2\} & \text{if } \tilde{\Sigma} \text{ is of type } B_n. \end{cases}$$

Hence the lemma is clear by the induction on N if $N' = 0$.

Suppose $N' > 0$. By the preceding lemma, we may assume that the fundamental system of Δ_1 is (4.22), (4.23) or (4.24). Then $\{\alpha \in \tilde{\Sigma};\, \langle e_1 - e_2, \alpha \rangle = 0\}$ is generated by

$$\begin{cases} \{e_{m+1} - e_{m+2}, \ldots, e_{n-1} - e_n\} & \text{if } \tilde{\Sigma} \text{ is of type } A_n, \\ \{e_{m+1} - e_{m+2}, \ldots, e_{n-1} - e_n,\, e_n\} & \text{if } \tilde{\Sigma} \text{ is of type } B_n \end{cases}$$

and the lemma is clear by the induction on N'. □

Remark 4.7. i) Fix $\alpha \in \Delta'$. Let $v \in \mathbb{R}^n$ with $\langle \alpha, v \rangle = 0$. Then $\partial_v u_\alpha(\langle \alpha, x \rangle) = 0$.

ii) If the rank of $\bar{\Delta}'$ equals one, $u_\alpha(t)$ for $\alpha \in \Delta'$ is any function.

iii) If the rank of $\bar{\Delta}'$ is larger than one, $U_\alpha(t)$ with $\alpha \in \Delta'$ are global meromorphic functions and therefore we may study $\{U_\alpha(t);\ \alpha \in \Delta'\}$ under the image of a transformation by the Weyl group.

iv) By the irreducible decomposition in Definition 4.2 our classification reduces to the case when P is irreducible.

Theorem 4.8. *Let Δ' be an irreducible component of Δ. Then the potential function $R_{\Delta'}(x) := \sum_{\alpha\in\Delta'} u_\alpha(\langle\alpha, x\rangle)$ is a transformation of a function in the following list with $m \geq 2$ under a map generated by the Weyl group, translations and scalings of the coordinates (cf. Lemma 3.5).*

Type A_1: *If the rank of $\bar{\Delta}'$ equals 1, $R_{\Delta'}(x)$ is an arbitrary function of $\langle\alpha, x\rangle$ with $\alpha \in \Delta'$. This solution is called trivial.*

Type B_2: *A standard transform of the function in the list* (Trig-B_2) – (Toda-$C_2^{(1)}$) *in §3 with replacing (x, y) by (x_1, x_2).*

(Trig-B_m): *Trigonometric potential of type B_m:*

$$\sum_{1\leq i<j\leq m} C_0\bigl(\sinh^{-2}\lambda(x_i + x_j) + \sinh^{-2}\lambda(x_i - x_j)\bigr)$$
$$+ \sum_{k=1}^{m}\bigl(C_1 \sinh^{-2} 2\lambda x_k + C_2 \sinh^{-2}\lambda x_k + C_3 \cosh 2\lambda x_k + C_4 \cosh 4\lambda x_k\bigr).$$

(Trig-A_{m-1}-bry): *Trigonometric potential of type A_{m-1} with boundary:*

$$\sum_{1\leq i<j\leq m} C_0 \sinh^{-2}\lambda(x_i - x_j)$$
$$+ \sum_{k=1}^{m}\bigl(C_1 e^{-2\lambda x_k} + C_2 e^{-4\lambda x_k} + C_3 e^{2\lambda x_k} + C_4 e^{4\lambda x_k}\bigr),$$

(Toda-$B_m^{(1)}$-bry): *Toda potential of type $B_m^{(1)}$ with boundary:*

$$\sum_{i=1}^{m-1} C_0 e^{-2\lambda(x_i - x_{i+1})} + C_0 e^{-2\lambda(x_{m-1}+x_m)} + C_1 e^{2\lambda x_1} + C_2 e^{4\lambda x_1}$$
$$+ C_3 \sinh^{-2}\lambda x_m + C_4 \sinh^{-2} 2\lambda x_m,$$

(Toda-$C_m^{(1)}$): *Toda potential of type $C_m^{(1)}$:*

$$\sum_{i=1}^{m-1} C_0 e^{-2\lambda(x_i - x_{i+1})} + C_1 e^{2\lambda x_1} + C_2 e^{4\lambda x_1} + C_3 e^{-2\lambda x_m} + C_4 e^{-4\lambda x_m},$$

(Toda-$D_m^{(1)}$-bry): *Toda potential of type $D_m^{(1)}$ with boundary:*

$$\sum_{i=1}^{m-1} C_0\bigl(e^{-2\lambda(x_i - x_{i+1})} + e^{-2\lambda(x_{m-1}+x_m)} + e^{2\lambda(x_1+x_2)}\bigr)$$
$$+ C_1 \sinh^{-2}\lambda x_m + C_2 \sinh^{-2} 2\lambda x_m + C_3 \sinh^{-2}\lambda x_1 + C_4 \sinh^{-2} 2\lambda x_1,$$

(Toda-$A_{m-1}^{(1)}$): *Toda potential of type $A_{m-1}^{(1)}$:*

$$\sum_{i=1}^{m-1} C_0 e^{-2\lambda(x_i - x_{i+1})} + C_0 e^{2\lambda(x_1 - x_m)}.$$

Definition 4.9. We define some potential functions as specializations of the above.

(Trig-A_{m-1}): Trigonometric potential of type A_{m-1} is (Trig-A_{m-1}-bry) with $C_1 = C_2 = C_3 = C_4 = 0$.

(Toda-$B_m^{(1)}$): Toda potential of type $B_m^{(1)}$ is (Toda-$B_m^{(1)}$-bry) with $C_3 = C_4 = 0$.

(Toda-$D_m^{(1)}$): Toda potential of type $D_m^{(1)}$ is (Toda-$D_m^{(1)}$-bry) with $C_1 = C_2 = C_3 = C_4 = 0$.

(Toda-D_m-bry): Toda potential of type D_m with boundary is (Toda-$B_m^{(1)}$-bry) with $C_1 = C_2 = 0$.

(Toda-A_{m-1}): Toda potential of type A_{m-1} is (Toda-$C_m^{(1)}$) with $C_1 = C_2 = C_3 = C_4 = 0$.

(Toda-BC_m): Toda potential of type B_m is (Toda-$C_m^{(1)}$) with $C_1 = C_2 = 0$.

(Toda-D_m): Toda potential of type D_m is (Toda-$B_m^{(1)}$-bry) with $C_1 = C_2 = C_3 = C_4 = 0$.

Proof of Proposition 4.8. We may assume that $\bar{\Delta}'$ is not of type B_2. Then Lemma 4.4 says that for any elements α and β of Δ'_L, there exists a sequence $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k = \beta$ such that $\langle \alpha_i, \alpha_{i+1} \rangle \neq 0$ and $\alpha_i \in \Delta'_L$ for $i = 1, \ldots, m-1$. Note that the number of elements of Δ'_L is larger than one. Fix $\alpha \in \Delta'$. Then lemma 4.1 assures that $U_\alpha(t) = Ct^r(1 - at^r)^{-1}$ with $a \in \mathbb{C}$.

Case 1: $U_\alpha(t) = Ct^r$.
Lemma 4.1 proves that $U_\beta(t) = C_\beta t^{2\epsilon(\beta)r}$ for $\beta \in \Delta'_L$. Here $\epsilon(\beta) = 1$ or -1. Then the set $S_L = \{\epsilon(\beta)\beta;\, \beta \in \Delta'_L\}$ satisfies the assumption of Lemma 4.5 and therefore we may assume that S_L equals (4.22), (4.24), (4.25) or (4.27) under the transformation of an element of the Weyl group, which correspond to (Toda-A_{m-1}), (Toda-D_m), (Toda-$A_{m-1}^{(1)}$) and (Toda-$D_m^{(1)}$), respectively, if $U_{e_i}(t) = 0$ for $i = 1, \ldots, m$. Suppose $U_{e_i}(t) \neq 0$ with a suitable i satisfying $1 \leq i \leq m$. Then Lemma 4.1 shows that one of the the following two cases occurs, from which the statement of the proposition is clear.

Case 1-1: $U_{e_i}(t) = C't^r(1 - at^r) + C''t^{2r}(1 - at^{2r})$ with $a \neq 0$.
We may assume $a = 1$ by a translation. Lemma 4.1 shows that then $U_{-e_i - e_{i+1}}(t) = U_{e_i - e_{i+1}}(t)$ and if $i > 1$ and $U_{e_{i-1} - e_i}(t) = U_{e_{i-1} + e_i}(t)$ if $i < m$. Therefore

$$\begin{cases} i = 1 \text{ and } S_L \text{ equals (4.27)} & \big(\Rightarrow \text{(Toda-}D_m^{(1)}\text{-bry)}\big) \\ \text{or} & \\ i = m \text{ and } S_L \text{ equals (4.24) or (4.27)} & \big(\Rightarrow \text{(Toda-}B_m^{(1)}\text{-bry) or (Toda-}D_m^{(1)}\text{-bry)}\big). \end{cases}$$

Case 1-2: $U_{e_i}(t) = C't^{\epsilon_i r} + C''t^{2\epsilon_i r}$ with $\epsilon_i = \pm 1$.
Lemma 4.1 says $\epsilon_i \langle e_i, \alpha \rangle \leq 0$ for $\alpha \in S_L$, only the following cases occur.

$$\begin{cases} i = 1 \text{ and } S_L \text{ equals (4.22) or (4.24)} & \big(\Rightarrow \text{(Toda-}C_m^{(1)}\text{) or (Toda-}B_m^{(1)}\text{-bry)}\big), \\ i = m \text{ and } S_L \text{ equals (4.22)} & \big(\Rightarrow \text{(Toda-}C_m^{(1)}\text{)}\big). \end{cases}$$

Case 2: $U_\alpha(t) = C_\alpha t^r(1 - a_\alpha t^r)^{-1}$ with $a_\alpha \neq 0$.
The argument just before Lemma 4.1 says that the condition $U_\beta(t) \neq 0$ and $U_\gamma(t) \neq 0$ with $|\alpha| = |\beta| = |\gamma|$ means $U_{w_\beta(\gamma)}(t) \neq 0$. Hence $\{\pm\beta;\, \beta \in \Delta'_L\}$ is a root system of type A_{m-1} or D_m. We may assume $a_\alpha = 1$ for its simple root α and hence C_α and a_α does not depend on $\alpha \in \Delta'_L$ because of (4.18) and Proposition 2.1.

Case 2-1: $\bar{\Delta}'_L$ is of type A.
Considering $\{U_{e_i + e_{i+1}}, U_{e_i - e_{i+1}}, U_{e_i}, U_{e_{i+1}}\}$, Theorem 3.9 shows $U_{e_i}(t) = U_{e_{i+1}}(t) = C_1 t^r + C_2 t^{2r} + C_3 t^{-r} + C_4 t^{-2r}$ and this case is reduced to (Trig-A_{m-1}-bry).

Case 2-2: $\bar{\Delta}'_L$ is of type D.
By the same consideration as in the previous case we may assume $U_{e_\nu}(t) = C_1 t^r(1-$

$t^r)^{-1} + C_2 t^{2r}(1-t^{2r})^{-1} + C_3(t^r - t^{-r}) + C_4(t^{2r} - t^{-2r})$ for $\nu = 1, \ldots, m$. Hence this case is reduced to (Trig-B_m). □

Corollary 4.10. *The non-trivial solutions in Proposition 4.8 which have regular singularity at the point $t = 0$ are in Corollary 3.10 or in the following list.*

(Trig-BC_m-reg)
$$\sum_{1\le i<j\le m} C_0\bigl(\sinh^{-2}\lambda(x_i+x_j) + \sinh^{-2}\lambda(x_i-x_j)\bigr) + \sum_{k=1}^{m}\bigl(C_1\sinh^{-2}2\lambda x_k + C_2\sinh^{-2}\lambda x_k\bigr),$$

(Trig-A_{m-1}-bry-reg)
$$\sum_{1\le i<j\le m} C_0\sinh^{-2}\lambda(x_i-x_j) + \sum_{k=1}^{m}\bigl(C_1 e^{-2\lambda x_k} + C_2 e^{-4\lambda x_k}\bigr),$$

(Toda-D_m-bry)
$$C_0\sum_{i=1}^{m-1} e^{-2\lambda(x_i-x_{i+1})} + C_0 e^{-2\lambda(x_{m-1}+x_m)} + C_3\sinh^{-2}\lambda x_m + C_4\sinh^{-2}2\lambda x_m,$$

(Toda-BC_m)
$$C_0\sum_{i=1}^{m-1} e^{-2\lambda(x_i-x_{i+1})} + C_3 e^{-2\lambda x_m} + C_4 e^{-4\lambda x_m}.$$

Remark 4.11. We have a natural compactification X of the space $\mathbb{C}^n$ of t so that for every $w \in W_{\Sigma(B_n)}$

$$s_j^w = e^{-(x'_j - x'_{j+1})} \quad (j = 1, \ldots, n-1), \quad s_n^w = e^{-x'_n} \quad \text{with } x' = wx$$

gives a local coordinate system of X and $t_j = s_j^e$ $(j = 1, \ldots, n)$. Then the non-trivial potential functions $R(x)$ we have obtained is meromorphic on X.

If $R(x)$ is holomorphic at $(s_1^w, \ldots, s_n^w) = 0$ for any $w \in W_{\Sigma(B_n)}$, $R(x)$ is said to have *regular singularity at every infinity*. In this case, our classification says that $R(x)$ is decomposed to the functions (Trig-BC_m-reg) and (Trig-A_m) which exactly corresponds to Heckman-Opdam's potential function of classical type. This gives a characterization of Heckman-Opdam's hypergeometric equations.

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